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A nonlinear functional for general scalar hyperbolic conservation laws

Hongxia Liu [a], Tong Yang [b,*]

[a] *Department of Mathematics, Jinan University, Guangzhou, PR China*
[b] *Department of Mathematics, City University of Hong Kong, Kowloon, Hong Kong*

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Abstract

A generalized entropy functional was introduced in [T.-P. Liu, T. Yang, A new entropy functional for scalar conservation laws, Comm. Pure Appl. Math. 52 (1999) 1427–1442] for the scalar hyperbolic conservation laws with convex flux function. This functional was crucially used in the functional approach to the L^1 stability study on the system of hyperbolic conservation laws when each characteristic field is either genuinely nonlinear or linearly degenerate. However, how to construct the generalized entropy functional for scalar conservation laws with general flux, and then how to apply the functional approach to the L^1 study on general systems are still open. In this paper, we construct a new nonlinear functional which gives some partial answer to this question and we expect the analysis will shed some light on the future investigation in this direction.

Keywords: General scalar conservation laws; Generalized entropy functional; Glimm's functional; L^1 estimates

1. Introduction

Consider the Cauchy problem for a scalar hyperbolic conservation law

$$\begin{cases} u_t + f(u)_x = 0, \\ u(x,0) = u_0(x), \end{cases} \tag{1.1}$$

* Corresponding author.
E-mail address: matyang@math.cityu.edu.hk (T. Yang).

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where $f(u) \in C^2(\mathbb{R})$ satisfying for all u under consideration. It is well known that even for smooth initial data, the solution usually blows up as the appearance of shock wave because of the nonlinearity in the flux function $f(u)$. Hence, we need to consider the weak solution to (1.1) satisfying

$$\int_0^\infty \int_{-\infty}^\infty \big(u(x,t)\phi_t(x,t) + f\big(u(x,t)\big)\phi_x(x,t)\big)\,dx\,dt + \int_{-\infty}^\infty u_0(x)\phi(x,0)\,dx = 0, \tag{1.2}$$

for every function $\phi(x,t) \in C_0^1(\mathbb{R}^2)$. As a consequence, a discontinuity (u_-, u_+) in the weak solution satisfies the Rankine–Hugoniot condition

$$s(u_+ - u_-) = f(u_+) - f(u_-), \tag{1.3}$$

with s being the propagation speed. For later presentation, we use

$$\sigma(\alpha) = \sigma(u_-, u_+) = \frac{f(u_-) - f(u_+)}{u_- - u_+}$$

to denote the "speed" of a discontinuity α with left state u_- and right state u_+. To choose the physical discontinuity, an entropy for general scalar conservation laws was introduced in [15]:

Definition 1.1. A discontinuity (u_-, u_+) is called an entropy shock if $\sigma(u_-, u) \geqslant \sigma(u_-, u_+)$ when $u_- > u_+$; while $\sigma(u_-, u) \leqslant \sigma(u_-, u_+)$ when $u_- < u_+$, for all u between u_- and u_+.

Under this entropy condition, the existence and L^1 contraction for scalar conservation laws have been well studied, cf. [5,14]. Moreover, if one takes any convex function of a solution u as an entropy, then its time decay rate is of the order of the total areas of the regions in the u–y plane bounded by the curve $y = f(u)$ and the straight line connecting two end states summed over all the entropy shock waves in the solution. Moreover, this kind of area is of the order of the bifurcation between the Hugoniot curve and rarefaction wave curve for systems when the scalar conservation law is defined along the rarefaction curve passing through a given state, cf. [12]. However, the convex entropy is not good enough to study the L^1 stability of entropy solution for systems even though it is good enough to study the L^1 perturbation of a solution around a constant state, cf. [9,12]. For this purpose, a generalized entropy functional was introduced in [13] for the scalar conservation laws with convex flux and this functional captures exactly the nonlinearity effect on the time evolution of solutions to the systems of conservation laws, cf. [3,10,11]. The main purpose for introducing this entropy functional is to show that

$$\begin{aligned}
&\int_0^\infty \int_{-\infty}^\infty \big(\big|u_x(x,t)\big| + \big|v_x(x,t)\big|\big)\big|u(x+,t) - v(x+,t)\big| \\
&\qquad \times \big(\big|\sigma\big(\delta u(x,t)\big) - \sigma\big(u(x+,t), v(x+,t)\big)\big| + \big|\sigma\big(\delta v(x,t)\big) - \sigma\big(v(x+,t), u(x+,t)\big)\big|\big)\,dx\,dt \\
&\quad \leqslant O(1)\big(\mathrm{T.V.}(u_0) + \mathrm{T.V.}(v_0)\big)\|u_0 - v_0\|_{L^1},
\end{aligned} \tag{1.4}$$

where $\delta u(x,t) = (u(x-,t), u(x+,t))$ is viewed as a wave with speed $\sigma(\delta u(x,t))$, T.V. denotes the total variation in x direction and $\|\cdot\|_{L^1}$ is the L^1-norm. Notice that when $u(x+,t) =$

$u(x-,t)$, the speed of the wave is the characteristic speed $f'(u(x\pm,t))$, and $u_x(x,t)$ is chosen as $u_x(x+,t)$ for any function of bounded variation. However, whether the generalized entropy functional exists for general scalar conservation laws is not known yet. As an attempt to this problem, we construct a nonlinear functional for general scalar conservation laws in this paper. Even though it does not lead to the desired estimate (1.4) in the L^1 estimation, it indeed gives a bound on the left-hand side of (1.4).

For later use, we review some known properties of the solutions to the general scalar conservation laws. Firstly, the solution operator of a scalar conservation law is L^1 contractive as stated in the following lemma, cf. [14].

Lemma 1.1. *Let u_i, $i=1,2$, be two solutions of* (1.1) *satisfying the entropy condition, then*

$$\left\|u_1(x,t)-u_2(x,t)\right\|_{L_1} \leqslant \left\|u_1(x,s)-u_2(x,s)\right\|_{L_1}, \quad \textit{for } s \leqslant t.$$

For the decay of the classical entropy of any weak solution of (1.1), the following lemma was proved in [12]. In particular, by choosing the convex entropy $\eta(u)=\frac{u^2}{2}$ with entropy flux $q(u)=\int^u sf'(s)\,ds$, we have the following entropy estimate.

Lemma 1.2. *Let $u(x,t)$ be a weak solution to the scalar conservation law* (1.1) *consisting countably many admissible shocks, denoted by $\{\alpha_i\}$. Then we have*

$$\frac{d}{dt}\int_{-\infty}^{\infty} u^2(x,t)\,dx = -2\sum_{\alpha_i} A(\alpha_i).$$

Here, for any admissible shock $\alpha=(u^-,u^+)$, $A(\alpha)$ denotes the area bounded by the curve $y=f(u)$ and the straight line segment connecting the end points $(u^-,f(u^-))$ and $(u^+,f(u^+))$ in the u–y plane, cf. Fig. 1.

Note that this entropy estimate is closely related to the bifurcation of the Hugoniot curve from the rarefaction wave curve in the general system. And the decrease of the Glimm's functional

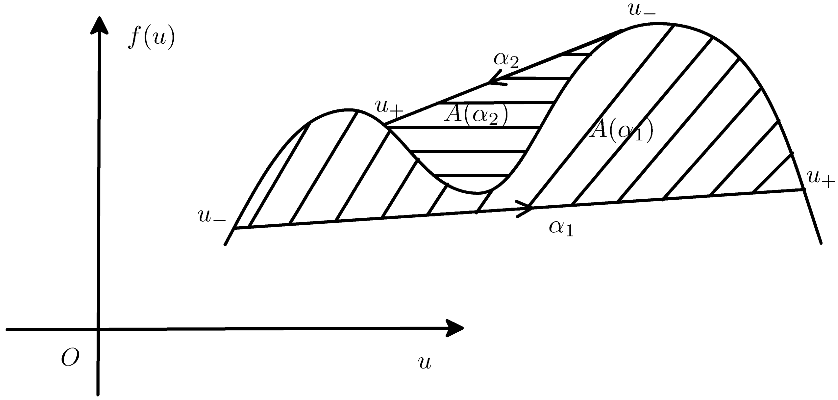

Fig. 1. $A(\alpha)$.

for the interaction of two shocks can also be well described by the subtraction of the area for the resulting shock from the areas of the shocks before interaction. That is, if the interaction of two shock waves α and β gives a larger shock wave γ, then we have

$$c_1|\alpha||\beta|\big|\sigma(\alpha)-\sigma(\beta)\big| \leqslant A(\gamma)-A(\alpha)-A(\beta) \leqslant c_2|\alpha||\beta|\big|\sigma(\alpha)-\sigma(\beta)\big|,$$

for some positive constants c_1 and c_2 depending on $\max_u\{|f'(u)|\}$ for all u under consideration, where $|\alpha||\beta||\sigma(\alpha)-\sigma(\beta)|$ is the decrease in the Glimm's functional through interaction. It is known that the quantity $|\alpha||\beta||\sigma(\alpha)-\sigma(\beta)|$ plays an important role in the study of hyperbolic conservation laws. In the following discussion, we will see that this quantity together with $A(\gamma)-A(\alpha)-A(\beta)$ are also defined for some virtual waves.

For later use, we use $A(u_1,u_2,u_3)$ to denote the area of the triangle bounded by the straight lines connecting $(u_i, f(u_i))$, $i=1,2,3$. Note that

$$\begin{aligned} &c_1|u_1-u_2||u_1-u_3|\big|\sigma(u_1,u_2)-\sigma(u_1,u_3)\big| \\ &\quad\leqslant A(u_1,u_2,u_3) \leqslant c_2|u_1-u_2||u_1-u_3|\big|\sigma(u_1,u_2)-\sigma(u_1,u_3)\big|. \end{aligned} \tag{1.5}$$

With this preparation, the construction of the nonlinear functional and the main results of this paper will be given in the next section. For the general theory of conservation laws, besides the references mentioned above, interested readers please refer to [1,2,4,6–8,16–19] and references therein.

2. Nonlinear functional and main results

In this section, we will construct the nonlinear functional for the time evolution of two solutions to the general scalar conservation laws. The weak solution is assumed to have small total variation bound and the existence of this kind of solution is well known which can be obtained by some constructive methods, such as Glimm scheme and wave front tracking method. Without any ambiguity and loss of generality, we sometimes assume the initial data is piecewise constant so that the solution contains finitely many discontinuities in the form of either entropy shocks or small rarefaction shocks in the approximate solution in the wave front tracking argument. Since the strength of the rarefaction shock is arbitrarily small and tends to zero, the limit of the approximate solutions is the unique entropy solution, cf. [2]. For brevity, we will not give the construction of the approximate solution for the existence proof and focus on the nonlinear functional in a modified L^1 space. Interested readers please refer to [2,4,17] for details of the existence and stability analysis.

Let $u(x,t)$ and $v(x,t)$ be two entropy solutions to the scalar conservation law (1.1) with small total variations. Moreover, we assume that the initial data satisfy $u(x,0)-v(x,0)=u_0(x)-v_0(x)\in L^1(\mathbb{R})$. In the following, we view $u_x(x,t)$ as a wave in $u(x,t)$ located at x and time t with strength $u(x+,t)-u(x-,t)$. We now first define the L^1 distance between $u(x,t)$ and $v(x,t)$ on the left or right with respect to the location x depending on the relative propagation speed between the wave located at x and the virtual wave $(u(x+,t), v(x+,t))$.

Set

$$L(u,v)(x,t) = \begin{cases} \int_x^\infty (u-v)_{\mathrm{sign}(u-v)(x+)}(y,t)\,dy + \int_\infty^x (u-v)_{-\mathrm{sign}(u-v)(x+)}(y,t)\,dy, \\ \quad \text{if } \sigma(u_x(x,t)) \geqslant \sigma(u(x+,t),v(x+,t)), \\ \int_x^\infty (u-v)_{-\mathrm{sign}(u-v)(x+)}(y,t)\,dy + \int_{-\infty}^x (u-v)_{\mathrm{sign}(u-v)(x+)}(y,t)\,dy, \\ \quad \text{if } \sigma(u_x(x,t)) < \sigma(u(x+,t),v(x+,t)), \end{cases} \tag{2.1}$$

where $f_\pm = |f|$ if $\pm f \geqslant 0$, otherwise it is zero.

Now the nonlinear functional can be defined by

$$\begin{aligned} E(u,v)(t) = {} & \int_{-\infty}^{\infty} A\big(u(x+,t),u(x-,t),v(x+,t)\big)L(u,v)(x,t)\,dx \\ & + \int_{-\infty}^{\infty} A\big(v(x+,t),v(x-,t),u(x+,t)\big)L(v,u)(x,t)\,dx \\ & + k\big(G(u)(t)+G(v)(t)\big)\int_{-\infty}^{\infty} \big|u(x,t)-v(x,t)\big|\,dx, \end{aligned} \tag{2.2}$$

where $k > 0$ is a constant chosen later. Here,

$$A\big(u(x+,t),u(x-,t),v(x+,t)\big) = \big|u_x(x,t)\big|\big|\sigma\big(u_x(x,t)\big) - \sigma\big(u(x\pm,t),v(x+,t)\big)\big|^2,$$

when $u(x,t)$ is differentiable at x, and $G(u)(t)$ is the Glimm's functional defined by

$$\begin{aligned} G(u)(t) = {} & \iint_{x<y} \big|u_x(x,t)\big|\big|u_x(y,t)\big|\big|\sigma\big(u_x(x,t)\big) - \sigma\big(u_x(y,t)\big)\big| \\ & \times \chi\big(\sigma\big(u_x(x,t)\big) - \sigma\big(u_x(y,t)\big)\big)\,dx\,dy, \end{aligned}$$

where $\chi(y) = 1$ for $y > 0$ and 0 otherwise.

The following theorem gives the decay estimate in time of the nonlinear functional $E(u,v)(t)$.

Theorem 2.1. *Let $u(x,t)$, $v(x,t)$ and $E(u,v)(t)$ be defined above. Then for k chosen suitably large, the nonlinear functional $E(u,v)(t)$ is decreasing in time, and except at the time of wave interaction it satisfies*

$$\begin{aligned} & \frac{d}{dt}E(u,v)(t) \\ & \quad \leqslant -c\int_{-\infty}^{\infty} \Big(\big|u_x(x,t)\big|\big|u(x+,t)-v(x+,t)\big|^2\big|\sigma\big(u_x(x,t)\big) - \sigma\big(u(x+,t),v(x+,t)\big)\big|^2 \\ & \qquad + \big|v_x(x,t)\big|\big|v(x+,t)-u(x+,t)\big|^2\big|\sigma\big(v_x(x,t)\big) - \sigma\big(v(x+,t),u(x+,t)\big)\big|^2\Big)\,dx. \end{aligned} \tag{2.3}$$

Here and in the sequel, $c > 0$ denotes a generic constant.

Proof. Without loss of generality, we can assume $u(x,t)$ and $v(x,t)$ are approximate solutions in the front tracking scheme so that they have finitely many wave fronts. Notice that for the exact weak solutions, the time evolution may contain infinitesimal cancellation between shock waves and expansion wave, the time derivative of the functional $E(u,v)$ may decrease faster than the estimation given for the approximate solutions, cf. [13] for the estimation on the generalized entropy functional for convex flux. However, since this kind of extra decreasing rate is not needed in the discussion of this paper, we do not include it for brevity. Except at the point of interaction, the nonlinear functional $E(u,v)(t)$ is then differentiable. Notice that through wave interaction, the first two terms in $E(u,v)(t)$ may increase as shown in the following two cases. However, the jump in these two terms can be compensated by the decrease of the Glimm's functional times the L^1 distance between the two solutions $u(x,t)$ and $v(x,t)$. For illustration, we consider the following two cases and other cases can be discussed similarly. In fact, in most of the cases of wave interaction, the first two terms in $E(u,v)(t)$ decrease or are unchanged.

Case 1. In the first case, we assume that at time t, the interaction of two shock waves α and β in the solution $u(x,t)$ gives a shock wave γ, and at the same time, the value of the other solution $v(x,t)$ at the location x_γ is the u coordinate at the point D, cf. Fig. 2. Here and in the sequel, x_α denotes the spatial location of the wave α. Notice that even though the sum of the strengths of α and β is the strength of γ, the value of $L(u,v)(x,t)$ has a jump at time t because of the change of the relative wave speeds with respect to the virtual wave (u_A,u_D). Here, u_A denotes the value of u of the point A in Fig. 2. And we use this kind of notations for the value of other points in the picture. To be precise, the change of the $E(u,v)(t)$ can be estimated as follows by assuming that there is only one wave interaction at time t without loss of generality, cf. Fig. 2:

$$
\begin{aligned}
&E(u,v)(t+) - E(u,v)(t-)\\
&\quad = A(u_A,u_C,u_D)L(u,v)(x_\gamma,t) - A(u_A,u_B,u_D)L(u,v)(x_\alpha,t)\\
&\qquad - A(u_B,u_C,u_D)L(u,v)(x_\beta,t) - k|\alpha||\beta|\bigl|\sigma(\alpha)-\sigma(\beta)\bigr| \int_{-\infty}^{\infty} \bigl|u(x,t)-v(x,t)\bigr|\,dx\\
&\quad \leqslant A(u_A,u_B,u_C)L(u,v)(x_\gamma,t) - k|\alpha||\beta|\bigl|\sigma(\alpha)-\sigma(\beta)\bigr| \int_{-\infty}^{\infty} \bigl|u(x,t)-v(x,t)\bigr|\,dx \leqslant 0,
\end{aligned}
$$

when k is chosen suitably large. Here, we have used the fact that $L(u,v)(x_\gamma,t)$ is bounded by the L^1 distance between two solutions and the decrease of the Glimm's functional through the interaction is given by $|\alpha||\beta||\sigma(\alpha)-\sigma(\beta)|$, cf. Fig. 2.

Case 2. As in Case 1, the interaction of the shock waves α and β also gives the shock wave γ, cf. Fig. 3. Now even though there is no change of $L(u,v)(x_\alpha,t)$ and $L(u,v)(x_\beta,t)$ before and after the interaction, the area in front of $L(u,v)(x_\gamma,t)$ is larger than the sum of the areas in front of $L(u,v)(x_\alpha,t)$ and $L(u,v)(x_\beta,t)$ by the amount of $A(u_A,u_B,u_C)$ which is the order of $|\alpha||\beta||\sigma(\alpha)-\sigma(\beta)|$. And this increase is then again compensated by the decrease of the Glimm's functional times the L^1 distance. We omit the similar calculation given for Case 1.

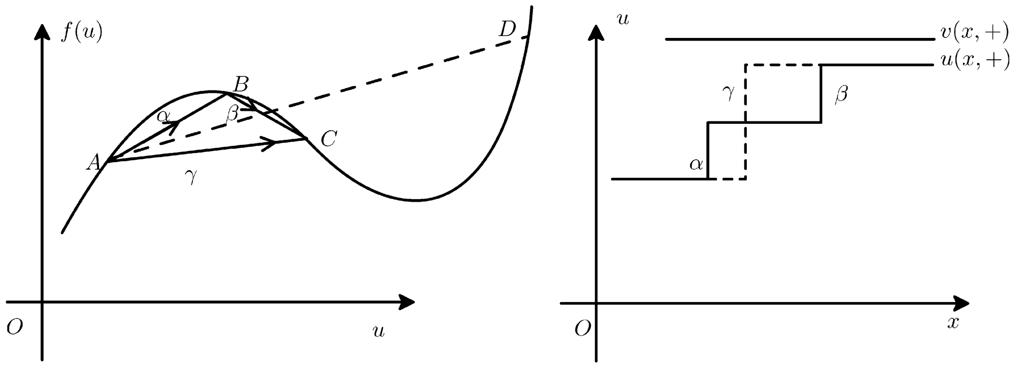

Fig. 2. Case 1.

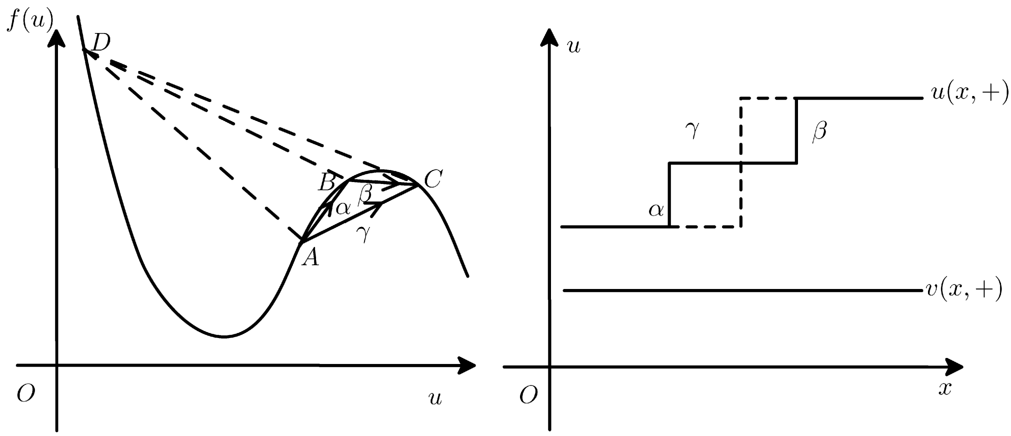

Fig. 3. Case 2.

Now we consider the situation when $E(u,v)(t)$ is differentiable. By the definition of $L(u,v)(x,t)$ and $L(v,u)(x,t)$ and the L^1 contraction property of the scalar conservation laws, it is straightforward to check that $E(u,v)(t)$ is decreasing in time. In fact,

$$\begin{cases} \dfrac{\partial}{\partial t}L(u,v)(x,t) \\ \quad = -\left|u(x+,t)-v(x+,t)\right|\left|\sigma\left(u_x(x,t)\right)-\sigma\left(u(x+,t),v(x+,t)\right)\right| \leqslant 0, \\ \dfrac{\partial}{\partial t}L(v,u)(x,t) \\ \quad = -\left|v(x+,t)-u(x+,t)\right|\left|\sigma\left(v_x(x,t)\right)-\sigma\left(v(x+,t),u(x+,t)\right)\right| \leqslant 0. \end{cases} \tag{2.4}$$

Furthermore, if there is no wave interaction at time t, then the time derivative of $G(u)(t)$, $G(v)(t)$ and $\|u-v\|_{L^1}(t)$ are all zero. Moreover, (1.5), (2.4) and the definition of $E(u,v)(t)$ imply (2.3) and this completes the proof of the theorem. □

The next theorem is about the estimation of the integral on the left-hand side of (1.4). For this, notice that there are only combination and cancellation of waves through the interaction of shock and shock, shock and rarefaction wave for scalar conservation laws. Therefore, any wave in time t can be traced back to time $s < t$ with corresponding wave strength by simple division if needed. In the following discussion, without any ambiguity, we view both solutions $u(x,t)$ and $v(x,t)$ as approximate solutions in the front tracking scheme so that they contain finitely many wave fronts and each wave at time $t > 0$ can be traced back to $t = 0$ with the same strength. Now let α be a wave located at x_α which is either an entropy shock wave or a small rarefaction shock in the solution $u(x,t)$. Define

$$E_\alpha(t) = A\big(u(x_\alpha+,t), u(x_\alpha-,t), v(x_\alpha+,t)\big)L(u,v)(x_\alpha,t) + kG_\alpha(t)\int_{-\infty}^{\infty}\big|u(x,t)-v(x,t)\big|\,dx, \tag{2.5}$$

where $G_\alpha(t)$ is the part of the terms in the Glimm's functional related to the wave α. That is

$$G_\alpha(t) = \sum_\beta |\alpha||\beta|\big|\sigma(\alpha)-\sigma(\beta)\big|\chi\big((x_\alpha - x_\beta)\big(\sigma(\beta)-\sigma(\alpha)\big)\big), \tag{2.6}$$

where β is any wave in the solution $u(x,t)$ at time t.

Corollary 2.1. *The nonlinear functional $E_\alpha(t)$ is decreasing in time and satisfies*

$$\frac{d}{dt}E_\alpha(t) \leqslant -c|\alpha|\big|u(x_\alpha+,t)-v(x_\alpha+,t)\big|^2\big|\sigma(\alpha)-\sigma\big(u(x_\alpha+,t), v(x_\alpha+,t)\big)\big|^2, \tag{2.7}$$

when it is differentiable, that is, at the time of no wave interaction. Here, c is a positive constant depending only on the flux function $f(u)$.

The proof of the corollary is similar to the one for Theorem 2.1 and we omit it for brevity.

Now we prove the main theorem in this paper about the estimation on the integral on the left-hand side of (1.4).

Theorem 2.2. *For the solutions $u(x,t)$ and $v(x,t)$ defined above, we have for any $t > 0$,*

$$\begin{aligned}&\int_0^t\int_{-\infty}^{\infty}\big(\big|u_x(x,t)\big| + \big|v_x(x,t)\big|\big)\big|u(x+,t)-v(x+,t)\big| \\ &\quad\times\big(\big|\sigma\big(\delta u(x,t)\big)-\sigma\big(u(x+,t),v(x+,t)\big)\big| + \big|\sigma\big(\delta v(x,t)\big)-\sigma\big(v(x+,t),u(x+,t)\big)\big|\big)\,dx\,dt \\ &\quad\leqslant ct^{\frac12}\big(\mathrm{T.V.}(u_0)+\mathrm{T.V.}(v_0)\big)^2\|u_0-v_0\|_{L^1}^{\frac12},\end{aligned} \tag{2.8}$$

for some uniform positive constant c.

Proof. By Corollary 2.1, we have for any wave α in either the solution $u(x,t)$ or $v(x,t)$,

$$
\begin{aligned}
&|\alpha| \int_0^t \left|u(x_\alpha+,t) - v(x_\alpha+,t)\right|^2 \left|\sigma(\alpha) - \sigma\big(u(x_\alpha+,t), v(x_\alpha+,t)\big)\right|^2 dt \\
&\quad \leqslant E_\alpha(0) \leqslant c|\alpha|\big(\mathrm{T.V.}(u_0) + \mathrm{T.V.}(v_0)\big)^2 \|u_0 - v_0\|_{L^1}.
\end{aligned}
$$

Here, we assume that the wave α survives from $t = 0$ to $t > 0$. If not, we can just adjust the time t to the maximum survival time of α without any extra difficulty. Thus

$$
\begin{aligned}
&\sum_\alpha |\alpha| \int_0^t \left|u(x_\alpha+,t) - v(x_\alpha+,t)\right| \left|\sigma(\alpha) - \sigma\big(u(x_\alpha+,t), v(x_\alpha+,t)\big)\right| dt \\
&\quad \leqslant \sum_\alpha \left(\int_0^t |\alpha|\, dt\right)^{\frac{1}{2}} \\
&\qquad \times \left(\int_0^t |\alpha| \left|u(x_\alpha+,t) - v(x_\alpha+,t)\right|^2 \left|\sigma(\alpha) - \sigma\big(u(x_\alpha+,t), v(x_\alpha+,t)\big)\right|^2 dt\right)^{\frac{1}{2}} \\
&\quad \leqslant c \sum_\alpha t^{\frac{1}{2}} |\alpha| \big(\mathrm{T.V.}(u_0) + \mathrm{T.V.}(v_0)\big) \|u_0 - v_0\|_{L^1}^{\frac{1}{2}} \\
&\quad \leqslant c t^{\frac{1}{2}} \big(\mathrm{T.V.}(u_0) + \mathrm{T.V.}(v_0)\big)^2 \|u_0 - v_0\|_{L^1}^{\frac{1}{2}},
\end{aligned}
$$

where the summation is over all waves α in the approximate solutions of $u(x,t)$ and $v(x,t)$. And this completes the proof of the theorem. □

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